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A brief methodological note on chaos theory and its recent applications based on new computer resources

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Abstract
Chaos theory refers to the behaviour of certain deterministic nonlinear dynamical systems whose solutions, although globally stable, are locally unstable. These chaotic systems describe aperiodic, irregular, apparently random and erratic trajectories, i.e., deterministic complex dynamics. One of the properties that derive from this local instability and that allow characterizing these deterministic chaotic systems is their high sensitivity to small changes in the initial conditions, which can be measured by using the so-called Lyapunov exponents. The detection of chaotic behaviour in the underlying generating process of a time series has important methodological implications. When chaotic behaviour is detected, then it can be concluded that the irregularity of the series is not necessarily random, but the result of some deterministic dynamic process. Then, even if such process is unknown, it will be possible to improve the predictability of the time series and even to control or stabilize the evolution of the time series. This article provides a summary of the main current concepts and methods for the detection of chaotic behaviour from time series.

Keywords: Chaos Theory, complex dynamics, detection of chaotic behaviour, Lyapunov exponents, nonlinear time series.
JEL Classification: C61, C22, C53, C54

Nota metodológica sobre la teoría del caos y las nuevas aplicaciones basadas en los recientes recursos computacionales

Resumen
La teoría del Caos se refiere al comportamiento que muestran ciertos sistemas dinámicos no lineales deterministas cuyas soluciones, aunque globalmente estables, resultan localmente inestables. Estos sistemas caóticos describen trayectorias aperiódicas, e irregulares, aparentemente aleatorias y erráticas, esto es, una dinámica compleja determinista. Una de las propiedades que se derivan de esa inestabilidad local y que permiten caracterizar a estos sistemas caóticos deterministas es su alta sensibilidad a los pequeños cambios en las condiciones iniciales, que puede medirse mediante el uso de los denominados exponentes de Lyapunov. La detección de comportamientos caóticos en el proceso subyacente generador de una serie temporal tiene importantes implicaciones metodológicas. Cuando se detecta comportamiento caótico, entonces se puede concluir que la irregularidad de la serie no es necesariamente aleatoria, sino el resultado de algún proceso dinámico determinista. Entonces, aunque dicho proceso sea desconocido, será posible mejorar las predicciones de la serie temporal e incluso controlar o estabilizar la evolución de dicha serie temporal. Este artículo proporciona un resumen de los principales conceptos y métodos actuales para la detección de comportamientos caóticos a partir de series temporales.

Palabras Clave: Teoria del Caos, Dinámica compleja, detección comportamiento caótico, exponentes Lyapunov, series de tiempo no lineales.
Clasificación JEL: C61, C22, C53, C54

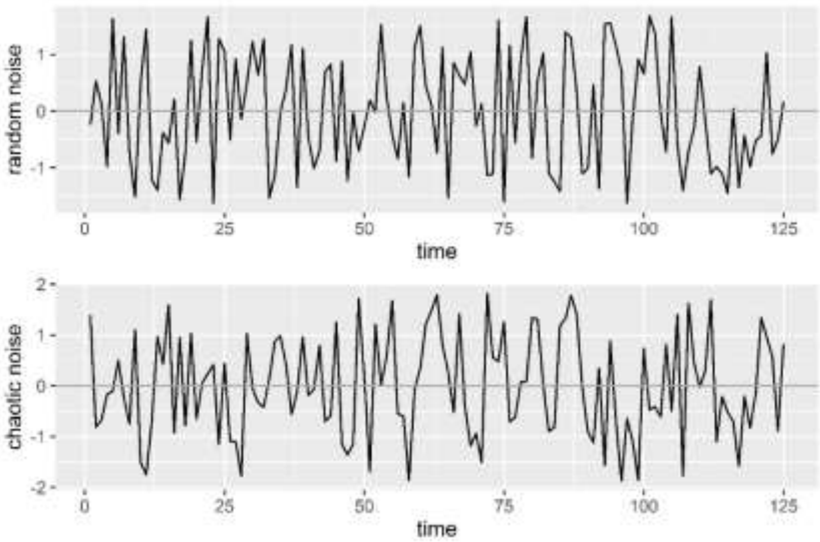
Introduction

Chaos theory has been considered as the third greatest discovery on Science after Relativity and Quantum Mechanics in the twentieth century. The term chaos has been hailed as a revolution of thoughts and attracting the ever increasing attention of many scientists from diverse disciplines including mathematics, statistics, data science, physics, cosmology, computation, engineering, chemistry, biology, medicine, neurology, psychology, economics and many others. It has become a truly multi-disciplinary area of research, and even has captured the imagination of the general public.

Traditionally the study of a phenomena with a complex and irregular evolution has been carried out assuming that the underlying dynamic, which generates this complexity, should be represented through stochastic processes. This approach has been propitiated, in part, because the solutions from deterministic systems (perfectly regular, ordered and periodic) were incapable of reproducing the complex dynamics observed in real phenomena. In fact, the maximum degree of complexity that deterministic systems could describe was restricted to quasi-periodic movements.

Nowadays, it is generally accepted that some simple deterministic dynamic systems can generate aperiodic, complex and irregular solutions. These dynamic systems are known as *chaotic systems*, that are nonlinear deterministic dynamic systems which can behave with an apparently erratic and irregular motion, see Fig 1.

Figure 1. Evolution of a pure random noise and a chaotic (deterministic) noise



In this sense a challenging question in chaos theory from a statistical perspective would be how to differentiate chaotic motions from purely random fluctuations. How to distinguish whether an apparently erratic, non-regular and aperiodic dynamic system is random or chaotic.

In order to be able to detect chaotic behavior, we must first be able to characterise a chaotic dynamic system in some way. We will dedicate the following section to this issue. On the other hand, it is also interesting to understand the epistemological implications of the detection of chaotic behavior in a system or a time series. We will devote the rest of this chapter to this other aspect.

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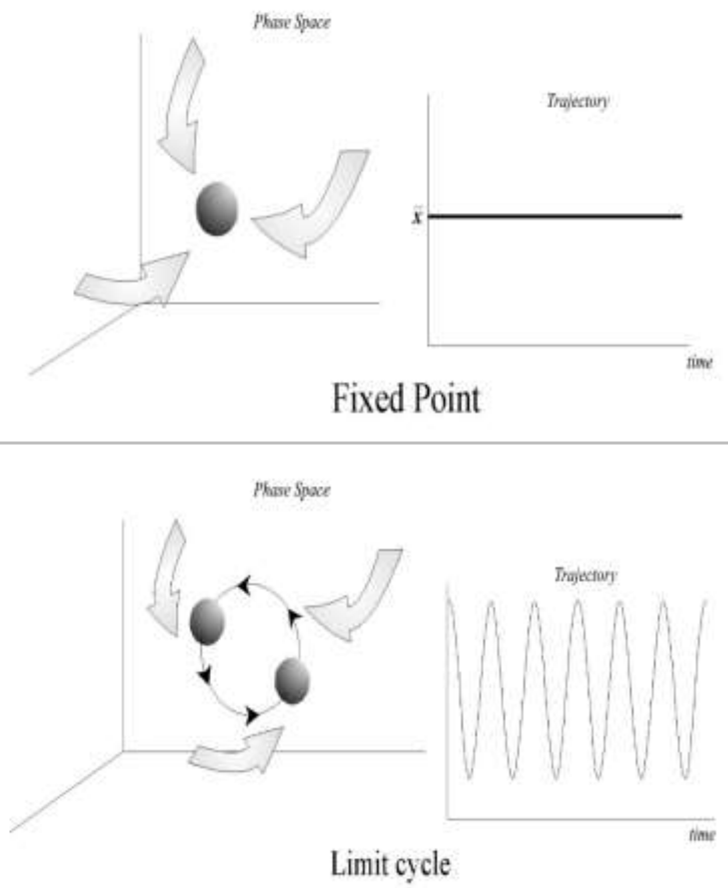
The pursuit of an operational definition of chaos

Generally speaking, there are several definitions of deterministic chaos that are commonly in use, for a review see e.g., *Li and Yorke (1975)*, *Katok (1980)*, *Berge (1984)*, *Ford (1986)*, *Shuster (1988)*, *Devaney (1989)*, *Ruelle (1993)* or *Brown (1996)*. As far as we know all definitions of chaos provided by those authors suffer from some defects but the most serious is that some definitions cannot be derived from each other. It is almost impossible to give a precise definition of chaos which at the same time encapsulates all that the term implies in the diverse literature. Therefore we will adopt an operational approach in order to characterise a chaotic dynamic system based on the study of its local stability, which plays a crucial role in the dynamics behind the system.

There are many approaches for the measurement and definition of the stability of a system in the literature. We will use the concept of local stability in a Lyapunov sense. This is a local definition of stability that measures the behaviour of a system inside its attractors. An attractor or attracting set is the dynamic equilibria of the system. In essence, an invariant set under the action of the dynamic system, non-reducible and with a basin of attraction. That is, the subset of the state space to which the trajectories of the system converge after a transient time, a region in which the system is trapped, without getting out of it except by exogenous perturbation, and a set that cannot be decomposed in other invariant disjunct subsets.

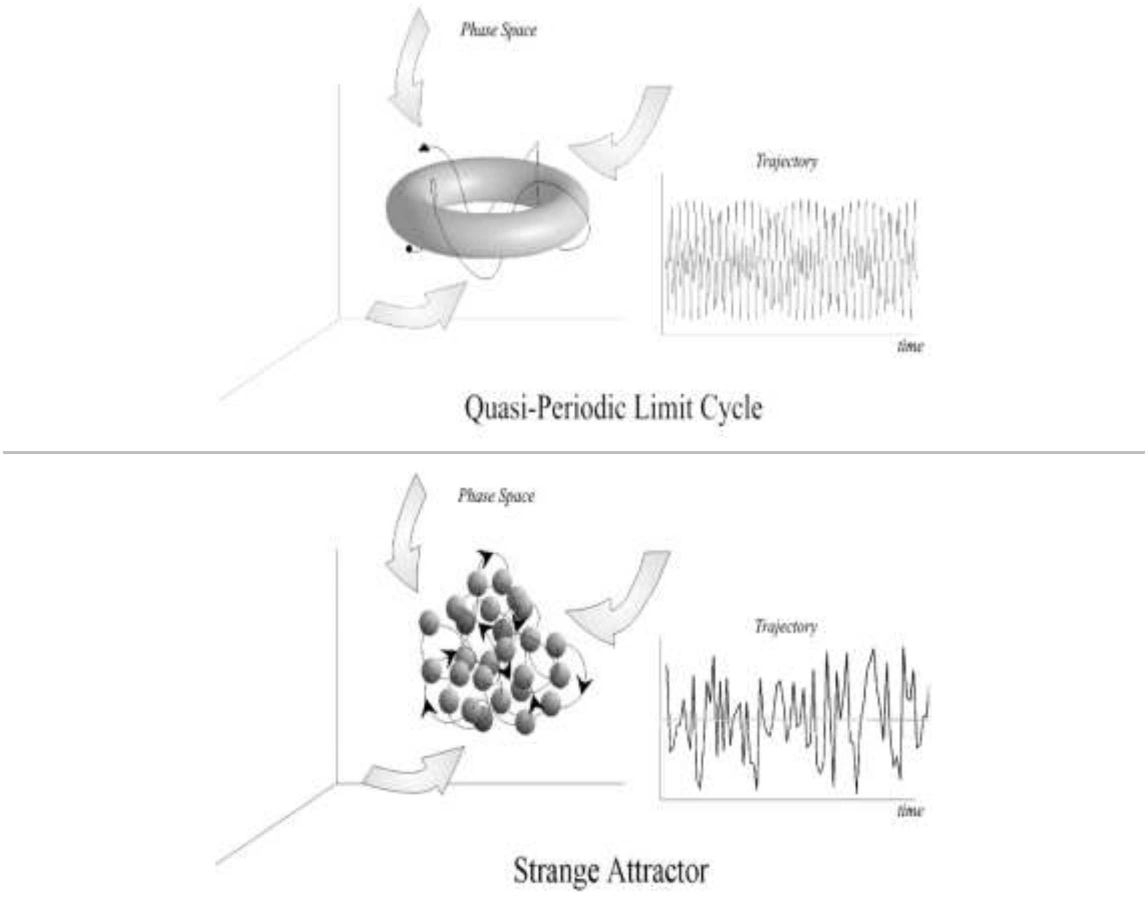
Fig 2. Illustration of simple and complex dynamics equilibria: Fixed point and limit cycle



Deterministic dynamical systems can have different types of dynamic equilibria, i.e. different types of attractors (see Fig 2 and Fig 3), mainly simple dynamics equilibria (fixed point; periodic points or limit cycle; quasi-periodic limit cycle or limit torus); and complex dynamics equilibria (strange or fractal attractors). Precisely, the dynamic simplicity or complexity of these equilibria is defined according to the stability of the trajectories within these attractors.

The stability of the system, following this 'local Lyapunov approach,' will be determined by the growth rate of divergence (or convergence) of two initially nearby trajectories. The smaller the magnitude of this divergence, the greater the simplicity of the system dynamics. In stable systems, with simple dynamics, two initially close points will always evolve close to each other. This type of stable system provides very accurate predictions of the future evolution or trajectories of the system.

Fig 3. Illustration of simple and complex dynamics equilibria: quai-periodic limite cycle and strange Attractors



On the contrary, some dissipative nonlinear deterministic system (chaotic), even though they have a global attractor, are locally unstable. A chaotic system with two initial values, no matter how close they are to each other, will lead to drastically different orbits or trajectories. In this sense, the dynamic equilibria cannot be a fixed point, periodic point, limit cycle, quasi-periodic limit cycle nor a limit torus. This local instability led to complex dynamics and to an operational definition of chaotic dynamics: the dynamics described by a (nonlinear) system that converge to a dynamic equilibria or attractor (a

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strange attractor), but whose trajectories within this attractor are unstable, irregular, aperiodic and complex.

This feature of chaos emergeS because the strange attractors are both a finite region of the state space but conformed by an infinite number of points (has a fractal structure). Points that the system sequentially visits, but without passing through the same point twice. That is, choatic dynamics describe trajectories with no regular finite period (or a period that tends to infinity) and it is highly unstable (for a review see Ruelle and Takens, 1971).

Thus, to distinguish simple dynamics from complex dynamics, that is, to distinguish periodic (or quasi-periodic) attractors from strange attractors, it is sufficient to analyse the instability of the system in its evolution within these dynamic equilibria. And to study this local stability we refer to the so-called *Lyapunov exponents*.

As mentioned above, the Lyapunov exponents measures how fast a perturbation in a state moves down the trajectory in a finite number of steps. That is, the Lyapunov exponents give (logarithm of) the average exponential rate of divergence of two infinitesimal nearby initial conditions on the attractor. There will be as many exponents (spectrum of exponents) as there are dimensions on the dynamic system. The sign of Lyapunov exponents (λ_i) measure the divergence ($\lambda_i > 0$) or convergence ($\lambda_i < 0$) of two initial infinitesimally nearby orbits in each of the dimension of the phase space.

This interpretation of the stability of the dynamic equilibria coincides exactly with the definition of *sensitivity to initial conditions*. Then, we can use the Lyapunov exponents for quantifying the initial-value sensitivity of two neighboring points. To have a chaotic behavior in a strange attractor, and *sensitivity to initial conditions,* it is necessary to have almost one positive Lyapunov exponent (for a review see *Gencay and Dechert, 1992).*

E.N. Lorenz was the first to show empirically this notion of initial-value sensitivity in his paper *Deterministic Non-periodic Flow (1963)*. In that paper he derived a nonlinear system for thermal convection in a simplified model of atmospheric flow and noticed a very strange behavior: the solutions of the equations could be unpredictable and irregular despite being deterministic. In particular, what he discovered was that if he stopped of iterate the solutions of his dynamic system and did it again start from a very similar initial condition, the solutions would immediately separate. It is written as a novel in his book *The Essence of Chaos (1994)*. The sensitive dependence of the evolution of a system for an infinitesimal change of initial conditions is called popularly the *butterfly effect*. Most authors have recognized this feature as the primary source of chaos.

The fact that a dynamic system shows sensitivity to initial conditions is a necessary but *not sufficient condition* for chaos *per se*. It is also necessary that the evolution of the solutions is bounded in some region of the state space. That is to say, the orbits must stay on an attractor. This last condition is always verified when the dynamic system is *dissipative*. In a *conservative* system, the volume of a given set is preserved in the finite-dimensional phase space under the action of the system over time while in a dissipative one, the system state shrinks or reduces asymptotically to a compact set. That is, it converges over time towards an attractor set. Hence in order to guarantee that the sensitive dependence on initial conditions property becomes *sufficient condition* for the existence of chaos, it is necessary that the evolution of the solutions is limited in some region (*global boundedness*).

To sum up, chaotic dynamical systems are those converging to locally unstable strange attractors having a high sensitivity to initial conditions. However, this is still some distance from a rigorous definition of chaos, and we are not aware of any universally agreed definition. As we said before it appears that for any definition of chaos, there may always be some chaotic systems which do not fall under some of them. Thus, making chaos a twin to Gödel's undecidability. So far as this dissertation is concerned, we have considered convenient and relevant to concentrate our attention on two features following *Giannerini (2002)*: a deterministic dynamical system is said to be *chaotic* if satisfies the following properties: (i) global boundedness (the dynamic system is *dissipative and converge to an attractor*); (ii) initial-value sensitivity in nearby trajectories, that is, at least one Lyapunov exponent is *positive* (the dynamic system converge a *strange attractor*).

Chaos detection from time-series data

Once we have established how to characterise a chaotic behaviour, we can proceed to present techniques to detect a chaotic signal from time-series data. According to the above definition of a chaotic system (it must be dissipative or convergent to an attractor, and with sensitivity to initial conditions), we can use Lyapunov exponents to check when a time series comes from a chaotic system. As the Lyapunov exponents (λ_i) measure the divergence ($\lambda_i > 0$) or convergence ($\lambda_i < 0$) of the system in each of the dimension of the phase space, we can use the sum of all the Lyapunov exponents spectrum values to check if the system is dissipative (when the sum is negative, ie. $\Sigma \lambda_i < 0$); and we can use the sign of the *largest* Lyapunov exponent to check for initial-value sensitivity ($\lambda_{max} > 0$).

As we have already mentioned, one of the properties of chaotic systems is that, although they are perfectly deterministic, they display complex dynamics. That is, they describe irregular, aperiodic and erratic behaviour, almost indistinguishable of a pure random stochastic process. Contrary to chaotic systems the processes that generate purely random trajectories are neither dissipative (they tend to fill the entire phase space) nor do they have any positive exponent. Then, we can also use the Lyapunov exponents to know whether an apparently random trajectory describing erratic and aperiodic behaviour, comes in fact from a purely stochastic system or whether, on the contrary, it comes from a system that, at least in part, presents deterministic chaotic behaviour ($\Sigma \lambda_i < 0$ and $\lambda_{max} > 0$).

Note, therefore, that the key to detecting chaotic behaviour, versus purely stochastic behaviour or versus a deterministic system of simple dynamics (or any combination of both), lies in the estimation of the Lyapunov exponents of the system. When a dynamic system is known we can directly calculate analytic (or computationally) the full spectrum of the Lyapunov exponent value using or simulating the dynamic system. However, when we do not know the analytical expression of the system that generates an observed time series, is it possible to estimate their Lyapunov exponents? that is, can chaos be detected when we assume that the true underlying dynamic system generating the observed time series is unknown?

The empirical analysis of chaotic dynamic systems is based on the study of *observed time series*. The main objective of this empirical analysis is precisely to obtain or to infer information about the properties of the data-generating process (deterministic or stochastic) that in most cases will be unknown. That is, the objective is to know whether the underlying generator system of a time-series data presents either a simple dynamics, behaves chaotically or comes from a purely stochastic process.

As the true data-generating process is unknown it is not possible to know the functional form that generates the dynamics associated with it. Instead, we assume the existence of an unknown *observer function* which transforms the unobserved state variable of the system into an observed time series data. Then it is assumed that all

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information available is a sequence of scalars, a univariate time series. As the true data-generating process is unknown, it is not possible to consider the true orbit of the dynamic system in the original state space. Nevertheless, we are going to be able to get an approximation (reconstruction) of the true underplayed unknown system that results equivalent in a topological sense. We mean equivalent in its dynamic and geometric properties (e.g., local stability of their attractors). This is an important result in chaos theory proposed by *Takens (1981)*.

This reconstruction procedure allows us to obtain all the relevant information about the unknown underlying dynamic system that generates the time-series data (*invariant properties*) like the Lyapunov exponents. Hence the Lyapunov exponents must have *approximately the same value* in both the true and the reconstructed state space. Then we can test the hypothesis of chaos in the unknown original dynamic system by using the Lyapunov exponents estimated with the reconstructed attractor.

Any method for estimating the Lyapunov exponent from time-series data are based on that *state space reconstruction* procedure. Let us explain briefly. The underlying idea in this reconstruction is to make copies of the single observable signal and consider those delayed values as coordinates of a reconstructed state space obtained from the time series. So, we must form a sequence of delayed coordinate embedding vectors[1].

Once we have reconstructed the attractor, we want to quantify the initial-value sensitive property estimating the Lyapunov exponents from time series in order to understand whether the unknown data-generating process shows a simple dynamic, behaved chaotically or resulted from a purely stochastic process.

There are two main methods in the literature that provide the estimated Lyapunov exponent from time-series data. The first, the so-called *direct approach* which measures the growth rate of the divergence between two trajectories with an infinitesimal difference in their initial conditions. The direct method was first proposed by *Wolf et al. (1985)*, and then revisited by *Rosenstein et al. (1993)*, and by *Kantz (1994)*. The underlying algorithm is explained in detail in *Kantz & Schreiber (1997)*.

The main drawbacks of these direct methods are the followings. First, it does not allow the estimation of the full spectrum of Lyapunov exponents just the largest, and this precludes testing the existence of an attractor ($\Sigma\ \lambda_i < 0$). Second, it is not robust to the presence of measurement noise. These direct estimators assign to chaos any divergence, even if purely random, caused by the measurement error itself and is therefore unable to distinguish when the irregular behaviour of a time series comes from purely random behaviour, and when it comes, at least in part, from a deterministic chaotic system. Third, it does not have a satisfactory performance in detecting existing nonlinearities on time-series of moderate sample sizes. Fourth, theoretical results for its consistency and asymptotic distributions are not available at the present time. This is a great disadvantage from a statistical perspective since there is then no possibility of making statistical inferences regarding chaos[2].

The second type of method used for estimating the Lyapunov exponents is the so-called *indirect or Jacobian approach*. These indirect methods solve all the disadvantages of the direct methods mentioned previously. The idea behind this approach can be

[1] More formally, let $\{x_t\}_{t=1}^{n}$ be the time-series data. We can perform the state space reconstruction of the underlying system using the method of delayed-coordinates proposed by Ruelle and Takens (1971). This procedure constructs a sequence on time *t* of delayed vectors in reconstructed state space $\mathbb{R}^m$:

$$\boldsymbol{x}_t^m = \left(x_t,\ x_{t-l},\ x_{t-2l},\ x_{t-3l}, \ldots,\ x_{t-(m-1)l}\right)$$

where *m* is the embedding dimension and *l* is the reconstruction time-delay (or lag).

[2] One way to overcome this problem is to make use of empirical approaches. In this sense, Giannerini and Rosa (2001) proposed a resampling scheme that allows us to get the confidence interval of the Lyapunov exponent estimator suggested by Kantz in a rigorous statistical way.

summarised briefly as follows. Rather than measuring directly the growth rate of the divergence between two nearby trajectories, these methods measure indirectly the average separation rate of two trajectories by estimating the derivative or Jacobian matrix associated with the reconstructed phase space from the time series. First, one must estimate the model that best approximates the dynamics of the system, and then calculate its partial derivatives to measure the instability of the system, i.e., to obtain the Lyapunov exponents.

This indirect methodology was first proposed by *Eckmann and Ruelle (1985)*, which is based on nonparametric regression methods. Further contributions focused on two different approaches. Firstly, those who used some lineal regressions, see e.g., *Sano and Sawada (1985)*, *Eckmann et al. (1986)*, *Brown et al. (1991)* or their extension in the form of polynomial regression proposed by *Lu and Smith (1997)*. The second approach was based on nonlinear regressions techniques following, see e.g., *Dechert and Gencay (1992), McCaffrey et al. (1992)*, *Nychka et al. (1992)*, *Whang and Linton (1999)*, *Shintani and Linton (2004)*[3].

For those interested in these techniques, and in the estimation of Lyapunov exponents the authors of this article have developed a novel R package called *DChaos* to test the hypothesis of chaotic behavior from a time series *(see Sandubete and Escot 2021).* This library provides an interface for researchers interested in the field of chaotic dynamic systems and nonlinear time series analysis and professors (and students) who teach (learn) courses related to those topics. The *DChaos* package contains some algorithms for detecting chaos from time-series data through the Lyapunov exponents by various computational methods based on the Jacobian indirect methods. It also allows making statistical inferences about its statistical significance, thus having a formal test to contrast the chaotic hypothesis from time series. These algorithms are publicly available at the Comprehensive R Archive Network https://cran.r-project.org/web/packages/DChaos/.

Implications of finding chaos in a time series.

The estimation of a positive exponent indicates that, although nonlinear and chaotic, the unknown generating process has a deterministic behavior, a time dependence, and a time feedback. This finding has three important implications in epistemological terms.

First, if we know that there is a unknown deterministic system that is generating the time series, as opposed to pure randomness, we should try to put all our efforts in finding out what exactly is this unknown generating system. And perhaps a necessary step in this direction would be to move away from the simplifying hypothesis of linearity and explore possibilities using nonlinear dynamic models.

But even if we are not able to know what this generating system might be, it is still possible to advance in two other important aspects: prediction and control of the system.

In fact, the second implication of detecting chaotic behavior in a time series is that there is room to try to improve their predictions, at least in the short term. The existence of a positive Lyapunov exponent reveals the existence of time dependence in the time

[3] This indirect approach states that exist a function $\boldsymbol{g}$: $\mathbf{R}^m \rightarrow \mathbf{R}^m$ such that $x_t^m = \boldsymbol{g}\,(x_{t-l}^m)$. Where x_t^m is the delayed reconstructed vectors. Under the assumption that the embedding is a homeomorphism, the map $\boldsymbol{g}$ is topologically conjugate to the unknown dynamic system $\boldsymbol{f}$, and then the estimation of Lyapunov exponents of $\boldsymbol{g}$ are equivalent to the underlaying of $\boldsymbol{f}$. The different indirect approach try to estimate $\boldsymbol{g}$ using different regressions approaches (see *Sandubete and Escot, 2021*).

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series, and therefore we can improve the predictive possibilities of the future by looking into the past, at least within the limits established by the sensitivity to the initial conditions. But to do this we have to move away from traditional (essentially linear) stochastic prediction models and use the techniques derived from chaos theory. These techniques do not require knowledge of the time series generating system. They are based on exploiting time dependence, recurrence and the existence of attractors in the reconstructed phase space (again using Takens' theorem). They do not try to fit a (linear) model to the data. Basically, these techniques search in the past for patterns similar to the present in order to predict the future. These predictions by analogues methods were proposed initially by Lorenz (1963, 1969) and developed by Farmer and Sidorowich (1987, 1988) and Casdagli (1989) who can be considered pioneers at the early stage of developments in predicting complex time series.

The third implication of chaos detection lies in the possibility of controlling or stabilizing the behavior of the time series. Again, a control that is performed even without determining the dynamical system that is generating the time series. Chaos control techniques, such as those initially proposed by Ott, Gregobi, and Yorke (1990) are based on the fact that when a nonlinear deterministic dynamical system exists, even if it is unknown, it can be forced towards one of its multiple stationary equilibria, even if these equilibria are initially unstable. Any fixed point can be reached without the need to force the system aggressively, simply by applying small perturbations that lead the system towards these stationary equilibria. With slight variations in the values of any control parameters (some parameter that one may discretionally alter) it is possible to change the dynamic equilibrium (the attractor), eliminating the irregularity of chaotic solutions, stabilizing irregular behaviors (when they are unwanted). This requires removing the system from the strange attractor, leading it to periodic equilibria (fixed points or limit cycles) and applying control rules to keep the system stabilized in those periodic equilibria. This type of control or stabilization would not be possible if the system were purely random, because it is based on the existence of a deterministic dynamic system, which, although unknown, is generating the observed time series.

Conclusion

Chaotic dynamical systems do not close the age old debate on determinism and indeterminism, which is still alive today. At best, it opens a new fuzzy boundary between the two paradigms. The theory or mathematics of chaos provides tools of analysis that situate us between formal determinism and asymptotic indeterminism. Formal determinism because chaotic systems are perfectly deterministic (but algebraically nonlinear) dynamical systems, which do not include any stochastic component or element in their formulation. And asymptotically indeterministic because in spite of not including any purely random effects, they present an irregular and aperiodic noisy behavior. This noisy chaos shows a high sensitivity to initial conditions (the so-called butterfly effect) that makes IT impossible to obtain tight predictions of the future state of the system beyond the short term. Indeed, even if we know the deterministic dynamical system generating a time series (which drives the evolution of the system in time), when there is chaotic behavior, small, very small variations in the initial point of prediction (due for example to measurement errors) mean that beyond the very short term the predictions are exponentially separated. Therefore, in practice, either one knows with infinite precision the true initial state of the system, or it is not possible to make accurate predictions of its future evolution.

Nevertheless, the detection of chaotic behavior in an observed time series has important implications in terms of predictability and stabilization of the time series generating system, even if it is unknown. When a time series presents irregular behavior because of deterministic chaotic behavior, then we can improve its prediction by exploiting time dependence, and we can control it in stable values by taking advantage of the existence of attractors or dynamic equilibria. And all this without needing to know the nature of the generating system.

In practice, most economic time series present an irregular cyclical behavior, and in fact, their origin is unknown. It is unknown what the real system generating the observed time series actually is. We do not know what is the real model that drives the evolution of the economy. It is crucial to know if this irregular behavior has a purely stochastic origin or if, on the contrary, it has a chaotic origin. Firstly, because the detection of chaotic behavior should make us rethink the use of linear modeling in most economic models. And secondly, because there are important implications in terms of prediction and control of the economy through new techniques of economic stabilization.

Chaos theory takes us into the complex terrain between formal determinism and practical indeterminism, opening a whole series of new possibilities and new methodologies. Many of these techniques, although initially proposed in the middle of the last century, have not been fully developed until the present time. These techniques are resource intensive for algorithmic computing. The recent development of new online parallel computing resources and Big Data techniques will undoubtedly provide great advances in this area in the very near future.

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